Kumaraswamy Weighted Exponential Distribution

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Abstract:
In this paper, a new extension distribution with four parameters named Kumaraswamy weighted exponential has been introduced based on the family of Kumaraswamy generalized distribution. The new density function can be expressed as an infinite linear combination of weighted exponential densities. Some of the mathematical properties, special cases along with the maximum likelihood estimations of the parameters of new distribution have been discussed.

توزيع كوماراسوامي الأسي الموزون

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المستخلص: في هذا البحث، تم تقديم توزيع كوماراسوامي الأسي الموزون كتوزيع جديد موسع بأربعة معلمات استنادا إلى عائلة التوزيع الموسع كوماراسوامي. يمكن صياغة دالة الكثافة الجديدة على أنها مزيج خطي غير منتهي من الكثافة الأسية الموزونة. تم مناقشة بعض الخصائص الرياضية ، الحالات الخاصة فضلا عن مقدرات الامكان الاعظم لمعلمات التوزيع الجديد.

1. Introduction

The weighted exponential (WE) distribution has been introduced by Gupta and Kundu in (2009) [3]. The WE distribution has received appreciable usage in the fields of engineering and medicine [7]. The probability density function (pdf) of WE distribution is given by [3]:

$$f_{WE}(x;\alpha,\lambda) = \frac{\alpha+1}{\alpha}\,\lambda\, e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right)\ ;\ x>0\ ;\ \alpha,\lambda>0 \qquad \dots(1)$$

The corresponding cumulative distribution function of WE distribution is given by [1]:

$$F_{WE}(x;\alpha,\lambda) = 1-\frac{1}{\alpha}\, e^{-\lambda x}\left(\alpha+1-e^{-\alpha\lambda x}\right) \qquad \dots(2)$$

The reliability and hazard functions of WE distribution at time (t), respectively, are given by [1]:

$$R_{WE}(t;\alpha,\lambda) = 1-F(t;\alpha,\lambda) = \frac{1}{\alpha}\, e^{-\lambda t}\left(\alpha+1-\ e^{-\alpha\lambda t}\right) \qquad \dots(3)$$

$$h_{WE}(t;\alpha,\lambda)=\frac{f(t;\alpha,\lambda)}{R(t;\alpha,\lambda)}=\frac{(\alpha+1)\,\lambda\left(1-e^{-\alpha\lambda t}\right)}{\alpha+1-e^{-\alpha\lambda t}} \qquad \dots(4)$$

The r^{th} moments about the origin is [4]:

$$E(X^r)=\frac{(\alpha+1)\,\Gamma(r+1)}{\alpha\,\lambda^r}\left(1-\frac{1}{(1+\alpha)^{r+1}}\right);\ r=1,2,3,\dots \qquad \dots(5)$$

The moment generating function, $M_X(t)$, for $-1<t<1$ can be expressed by [5]:

$$M_X(t)=E(e^{tX})=\frac{(\alpha+1)\lambda}{\alpha}\left[\frac{1}{\lambda-t}-\frac{1}{\lambda-t+\alpha\lambda}\right] \qquad \dots(6)$$

In particular [5]:

$$M'_X(0)=E(X)=\frac{\alpha+2}{\lambda(\alpha+1)} \qquad \dots(7)$$

$$M''_X(0)=E(X^2)=\frac{2(\alpha^2+3\alpha+3)}{\lambda^2(\alpha+1)^2} \qquad \dots(8)$$

Then,

$$v(X)=E(X^2)-[E(X)]^2=\frac{1}{\lambda^2}\left[1+\frac{1}{(\alpha+1)^2}\right] \qquad \dots(9)$$

2. Kumaraswamy Weighted Exponential (KWE) Distribution

For any baseline cumulative distribution function $G(x)$ of a random variable X with two additional shape parameters, $a,b>0$, Cordeiro and de Castro in (2011) [2] proposed Kumaraswamy generalized (KG) distribution with cumulative distribution, probability density, reliability and hazard functions given, respectively, by:

$$F_{KG}(x;a,b)=1-\left[1-\left(G(x)\right)^a\right]^b \qquad \dots(10)$$

$$f_{KG}(x;a,b)=abg(x)[G(x)]^{a-1}\left[1-\left(G(x)\right)^a\right]^{b-1} \qquad \dots(11)$$

$$R_{KG}(x;a,b)=\left[1-\left(G(x)\right)^a\right]^b \qquad \dots(12)$$

$$h_{KG}(x;a,b)=\frac{abg(x)\left(G(x)\right)^{a-1}}{1-\left(G(x)\right)^a} \qquad \dots(13)$$

where $g(x)=\frac{\partial G(x)}{\partial x}$. Hence, each new KG distribution can be generated from a specified cumulative distribution.

Now, suppose that $G(x)$ represents the WE cumulative distribution as in equation (2), then (10) and (11) yields (KWE) cumulative distribution and probability density functions for $x>0$, respectively, as:

$$F_{KWE}(x;\alpha,\lambda,a,b)=1-\left[1-\left(1-\frac{1}{\alpha}e^{-\lambda x}\left(\alpha+1-e^{-\lambda\alpha x}\right)\right)^a\right]^b \qquad \dots(14)$$

$$f_{KWE}(x;\alpha,\lambda,a,b)=ab\,\frac{\alpha+1}{\alpha}\lambda\,e^{-\lambda x}\left(1-e^{-\lambda\alpha x}\right)\left[1-\frac{1}{\alpha}e^{-\lambda x}\left(\alpha+1-e^{-\lambda\alpha x}\right)\right]^{a-1}\left[1-\left(1-\frac{1}{\alpha}e^{-\lambda x}\left(\alpha+1-e^{-\lambda\alpha x}\right)\right)^a\right]^{b-1} \qquad \dots(15)$$

where $\lambda>0$ is the scale parameter and $\alpha,a,b>0$ are the shape parameters. As special cases, when $a=b=1$, the probability density function for KWE

distribution will be the probability density function of WE and when $b = 1$, the probability density function for KWE distribution will be the probability density function of exponentiated WE distribution as in [6].

The reliability and hazard functions of KWE distribution at time (t) can be expressed as:

$$R_{KWE}(t;\alpha,\lambda,a,b) = \left[1-\left(1-\frac{1}{\alpha}e^{-\lambda t}\left(\alpha+1-e^{-\alpha\lambda t}\right)\right)^{a}\right]^{b} \quad \dots(16)$$

$$h_{KWE}(t;\alpha,\lambda,a,b) = \frac{ab\,\frac{\alpha+1}{\alpha}\lambda\, e^{-\lambda t}\left(1-e^{-\alpha\lambda t}\right)\left(1-\frac{1}{\alpha}e^{-\lambda t}\left(\alpha+1-e^{-\alpha\lambda t}\right)\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda t}\left(\alpha+1-e^{-\alpha\lambda t}\right)\right)^{a}} \dots(17)$$

3. Expansions for Cumulative and Density Functions of KWE Distribution

The cumulative and density functions of KWE distribution, equations (14) and (15), can be expansions according to the generalized binomial theorem, $(1+c)^{v} = \sum_{i=0}^{\infty}\binom{v}{i}\, c^{i}$, respectively as:

$$F_{KWE}(x;\alpha,\lambda,a,b) = 1-\sum_{i=0}^{\infty}(-1)^{i}\binom{b}{i}\left[1-\frac{1}{\alpha}\, e^{-\lambda x}\left(\alpha+1-e^{-\alpha\lambda x}\right)\right]^{ai}$$

$$f_{KWE}(x;\alpha,\lambda,a,b) = ab\,\frac{\alpha+1}{\alpha}\,\lambda\, e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right)\sum_{i=0}^{\infty}(-1)^{i}\binom{b-1}{i}\left(1-\frac{1}{\alpha}\, e^{-\lambda x}\left(\alpha+1-e^{-\alpha\lambda x}\right)\right)^{a(i+1)-1}$$

Now, suppose that:

$$\eta_i = (-1)^{i}\binom{b}{i} \quad ; \quad \binom{b}{i} = \frac{b(b-1)\dots(b-i+1)}{i!}$$

and,

$$w_j = \frac{ab}{j+1}\sum_{i=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}$$

Then the cumulative and density functions of KWE distribution can be expansions, respectively, as:

$$F_{KWE}(x;\alpha,\lambda,a,b) = 1-\sum_{i=0}^{\infty}\eta_i\; K(x;\alpha,\lambda,ai) \quad \dots(18)$$

$$f_{KWE}(x;\alpha,\lambda,a,b) = \sum_{j=0}^{\infty} w_j\; S(x;\alpha(j+1),\lambda) \quad \dots(19)$$

where,

$K(x;\alpha,\lambda,\theta) = \left[1-\frac{1}{\alpha}\, e^{-\lambda x}\left(\alpha+1-e^{-\alpha\lambda x}\right)\right]^{\theta}$ which can denotes the expansion WE cumulative distribution with parameters α,λ and $\theta = ai$.

$S(x;\alpha(j+1),\lambda)$ denotes the expansion WE density function with parameters $\alpha(j+1)$ and λ and cumulative distribution as in (2).

Thus, KWE density function can be expressed as an infinite linear combination of WE densities.

4. Moments and Moment Generating Function of KWE Distribution

The r^{th} moments about the origin can be expressed by:

$$E(X^r) = \sum_{j=0}^{\infty} w_j \int_0^{\infty} x^r\, S(x; \alpha(j+1), \lambda) dx$$

$$= \sum_{j=0}^{\infty} w_j \int_0^{\infty} x^r\, \frac{\alpha(j+1)+1}{\alpha(j+1)}\, \lambda\, e^{-\lambda x}\left(1 - e^{-\alpha(j+1)\lambda x}\right) dx$$

$$\Rightarrow E(X^r) = \sum_{j=0}^{\infty} w_j\, \lambda\, \frac{\alpha(j+1)+1}{\alpha(j+1)} \left[\frac{\Gamma(r+1)}{\lambda^{r+1}} - \frac{\Gamma(r+1)}{\left[\lambda\left(\alpha(j+1)\right)+1\right]^{r+1}}\right] \quad \ldots (20)$$

Now, setting $r = 1$ and $r = 2$, we get

$$E(X) = \sum_{j=0}^{\infty} w_j\, \lambda \frac{\alpha(j+1)+1}{\alpha(j+1)} \left[\frac{\Gamma(2)}{\lambda^2} - \frac{\Gamma(2)}{[\lambda(\alpha(j+1)+1)]^2}\right]$$

$$E(X^2) = \sum_{j=0}^{\infty} w_j\, \lambda \frac{\alpha(j+1)+1}{\alpha(j+1)} \left[\frac{\Gamma(3)}{\lambda^3} - \frac{\Gamma(3)}{[\lambda(\alpha(j+1)+1)]^3}\right]$$

Setting $a = b = 1$, we have,

$$w_j = \begin{cases} 1 & ; \quad j = 0 \\ 0 & ; \quad j \geq 1 \end{cases} \quad \ldots (21)$$

Hence, the mean and variance of X are given by:

$$\mu = E(X) = \lambda \frac{\alpha+1}{\alpha}\left[\frac{1}{\lambda^2} - \frac{1}{[\lambda(\alpha+1)]^2}\right] = \frac{\alpha+2}{\lambda(\alpha+1)}$$

$$v(X) = E(X^2) - [E(X)]^2 = \frac{2(\alpha^2+3\alpha+3)}{\lambda^2(\alpha+1)^2} - \frac{(\alpha+2)^2}{\lambda^2(\alpha+1)^2} = \frac{1}{\lambda^2}\left[1 + \frac{1}{(\alpha+1)^2}\right]$$

which are precisely the mean and variance of WE distribution.

The moment generating function of KWE distribution for $-1 < t < 1$ can be expressed by:

$$M_X(t) = E(e^{xt}) = \sum_{j=0}^{\infty} w_j \int_0^{\infty} e^{xt}\, S(x; \alpha(j+1), \lambda) dx$$

$$= \sum_{j=0}^{\infty} w_j \int_0^{\infty} e^{xt}\, \frac{\alpha(j+1)+1}{\alpha(j+1)}\, \lambda\, e^{-\lambda x}\left(1 - e^{-\alpha(j+1)\lambda x}\right) dx$$

$$\Rightarrow M_X(t) = \sum_{j=0}^{\infty} w_j\, \lambda\, \frac{\alpha(j+1)+1}{\alpha(j+1)} \left[\frac{1}{\lambda - t} - \frac{1}{\lambda - t + \lambda\alpha(j+1)}\right] \quad \ldots (22)$$

By (21),

$$M_X(t) = \frac{(\alpha+1)\lambda}{\alpha}\left[\frac{1}{\lambda - t} - \frac{1}{\lambda - t + \lambda\alpha}\right]$$

which is precisely the moment generating function of WE distribution.

5. Likelihood Function and Estimation

The maximum likelihood estimations (MLEs) of α, λ, a and b are the solution of the first partial derivatives of the natural-log likelihood function ℓ_{KWE} with respect to that parameters where the likelihood and natural-log likelihood functions for equation (15) are defined, respectively, by:

$$L(\alpha,\lambda,a,b|\underline{x}) = a^n b^n \frac{(\alpha+1)^n}{\alpha^n}\lambda^n e^{-\lambda\sum_{i=1}^n x_i}\prod_{i=1}^n\left(1-e^{-\lambda\alpha x_i}\right)\left[1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right]^{a-1}\left[1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^a\right]^{b-1} \quad \dots(23)$$

$$\ell_{KWE} = \ln L(\alpha,\lambda,a,b|\underline{x}) = n\ln a + n\ln b + n\ln(\alpha+1) - n\ln\alpha + n\ln\lambda - \lambda\sum_{i=1}^n x_i + \sum_{i=1}^n \ln\left(1-e^{-\lambda\alpha x_i}\right) + (a-1)\sum_{i=1}^n \ln\left[1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right] + (b-1)\sum_{i=1}^n \ln\left[1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^a\right] \quad \dots(24)$$

Now, since there are no closed forms of the solutions, Newton–Raphson iterative technique, can be used to obtain the MLEs as,

$$\begin{bmatrix}\hat{\alpha}\\ \hat{\lambda}\\ \hat{a}\\ \hat{b}\end{bmatrix}^{(h+1)} = \begin{bmatrix}\hat{\alpha}\\ \hat{\lambda}\\ \hat{a}\\ \hat{b}\end{bmatrix}^{(h)} - \mathrm{J}_{(h)}^{-1}\begin{bmatrix}\frac{\partial\ell_{KWE}}{\partial\alpha}\\ \frac{\partial\ell_{KWE}}{\partial\lambda}\\ \frac{\partial\ell_{KWE}}{\partial a}\\ \frac{\partial\ell_{KWE}}{\partial b}\end{bmatrix}^{(h)} \quad ;\ h = 0,1,2,\dots$$

where,

$$J_{(h)}=\begin{bmatrix}\frac{\partial^2\ell_{KWE}}{\partial\alpha^2} & \frac{\partial^2\ell_{KWE}}{\partial\alpha\,\partial\lambda} & \frac{\partial^2\ell_{KWE}}{\partial\alpha\,\partial a} & \frac{\partial^2\ell_{KWE}}{\partial\alpha\,\partial b}\\ \frac{\partial^2\ell_{WE-K}}{\partial\lambda\,\partial\alpha} & \frac{\partial^2\ell_{KWE}}{\partial\lambda^2} & \frac{\partial^2\ell_{KWE}}{\partial\lambda\,\partial a} & \frac{\partial^2\ell_{KWE}}{\partial\lambda\,\partial b}\\ \frac{\partial^2\ell_{KWE}}{\partial a\,\partial\alpha} & \frac{\partial^2\ell_{KWE}}{\partial a\,\partial\lambda} & \frac{\partial^2\ell_{KWE}}{\partial a^2} & \frac{\partial^2\ell_{KWE}}{\partial a\,\partial b}\\ \frac{\partial^2\ell_{KWE}}{\partial b\,\partial\alpha} & \frac{\partial^2\ell_{KWE}}{\partial b\,\partial\lambda} & \frac{\partial^2\ell_{KWE}}{\partial b\,\partial a} & \frac{\partial^2\ell_{KWE}}{\partial b^2}\end{bmatrix}^{(h)}$$

$$\frac{\partial\ell_{KWE}}{\partial\alpha}=\frac{n}{\alpha+1}-\frac{n}{\alpha}+\sum_{i=1}^{n}\frac{\lambda x_i\,e^{-\lambda\alpha x_i}}{1-e^{-\lambda\alpha x_i}}+\sum_{i=1}^{n}\frac{1}{\alpha}\,e^{-\lambda\,x_i}\left(\frac{1}{\alpha}-e^{-\lambda\alpha x_i}\left(\frac{1}{\alpha}+\lambda x_i\right)\right)\left[\frac{a-1}{1-\frac{1}{\alpha}e^{-\lambda\,x_i}\,(\alpha+1-e^{-\lambda\alpha x_i})}-\frac{a(b-1)\left(1-\frac{1}{\alpha}e^{-\lambda\,x_i}\,(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda\,x_i}\,(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}}\right]$$

$$\frac{\partial\ell_{KWE}}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{\alpha x_i\,e^{-\lambda\alpha\,x_i}}{1-e^{-\lambda\alpha\,x_i}}+\sum_{i=1}^{n}x_i\,e^{-\lambda x_i}\left(1+\frac{1}{\alpha}-e^{-\lambda\alpha x_i}\left(1+\frac{1}{\alpha}\right)\right)\left[\frac{a-1}{1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})}-\frac{a(b-1)\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\,(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}}\right]$$

$$\frac{\partial\ell_{KWE}}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}ln\left(1-\frac{1}{\alpha}\,e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)-(b-1)\sum_{i=1}^{n}\frac{\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}ln\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}}$$

$$\frac{\partial\,\ell_{KWE}}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}ln\left[1-\left(1-\frac{1}{\alpha}\,e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^{a}\right]$$

$$\frac{\partial^2\ell_{KWE}}{\partial\alpha^2}=\frac{n}{\alpha^2}-\frac{n}{(\alpha+1)^2}-\sum_{i=1}^{n}\frac{\lambda^2x_i^2\,e^{-\lambda\alpha x_i}}{1-e^{-\lambda\alpha x_i}}\left(1+\frac{e^{-\lambda\alpha x_i}}{1-e^{-\lambda\alpha x_i}}\right)-\sum_{i=1}^{n}\left(\frac{1}{\alpha}\,e^{-\lambda\,x_i}\left(\frac{1}{\alpha}-e^{-\lambda\alpha\,x_i}\left(\frac{1}{\alpha}+\lambda x_i\right)\right)\right)^2\left[\frac{a-1}{\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^2}+\frac{a(b-1)\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\,(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}}\left(\frac{a\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\,(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}}+\frac{a-1}{1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})}\right)\right]+$$

$$\sum_{i=1}^{n}\left[\frac{1}{\alpha^3}e^{-\lambda x_i}\left(2e^{-\lambda\alpha x_i}-2\right)-\frac{\lambda^2x_i^2}{\alpha}\,e^{-\lambda x_i(\alpha+1)}\right]\left[\frac{a-1}{1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})}-\frac{a(b-1)\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\,(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}}\right]$$

$$\frac{\partial^2 \ell_{KWE}}{\partial \lambda^2} = -\frac{n}{\lambda^2} - \sum_{i=1}^{n} \frac{\alpha^2 x_i^2\, e^{-\lambda\alpha x_i}}{1-e^{-\lambda\alpha x_i}}\left(1+\frac{e^{-\lambda\alpha x_i}}{1-e^{-\lambda\alpha x_i}}\right) - \sum_{i=1}^{n}\left(x_i\, e^{-\lambda x_i}\left(1+\frac{1}{\alpha} - e^{-\lambda\alpha x_i}\left(1+\frac{1}{\alpha}\right)\right)\right)^2 \left[\frac{a-1}{\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^2} + \frac{a(b-1)\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}}\left(\frac{a\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}} + \frac{a-1}{1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})}\right)\right] + \sum_{i=1}^{n}\left(x_i^2\, e^{-\lambda x_i(\alpha+1)}\left(\alpha+2+\frac{1}{\alpha}\right) - x_i^2\, e^{-\lambda x_i}\left(1+\frac{1}{\alpha}\right)\right)\left[\frac{a-1}{1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})} - \frac{a(b-1)\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}}\right]$$

$$\frac{\partial^2 \ell_{KWE}}{\partial a^2} = -\frac{n}{a^2} - \sum_{i=1}^{n} \frac{(b-1)\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}\left(ln\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}\right)^2}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}}\left[1+\frac{\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}}\right]$$

$$\frac{\partial^2 \ell_{KWE}}{\partial b^2} = -\frac{n}{b^2}$$

$$\frac{\partial^2 \ell_{KWE}}{\partial \alpha\, \partial \lambda} = \frac{\partial^2 \ell_{KWE}}{\partial \lambda\, \partial \alpha} = \sum_{i=1}^{n}\frac{x_i\, e^{-\lambda\alpha x_i}}{1-e^{-\lambda\alpha x_i}}\left[1-\alpha\lambda x_i - \frac{\alpha\lambda x_i\, e^{-\alpha\lambda x_i}}{1-e^{-\lambda\alpha x_i}}\right] + \sum_{i=1}^{n}\frac{x_i}{\alpha}\, e^{-2\lambda x_i}\left(\frac{1}{\alpha} - e^{-\lambda\alpha x_i}\left(\frac{1}{\alpha}+\lambda x_i\right)\right)\left(1+\frac{1}{\alpha} - e^{-\lambda\alpha x_i}\left(1+\frac{1}{\alpha}\right)\right)\left[\frac{-(a-1)}{\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^2} - \frac{a(b-1)\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}}\left(\frac{a-1}{1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})} + \frac{a\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}}\right)\right] + \sum_{i=1}^{n}\left[\lambda x_i^2\, e^{-\lambda x_i(\alpha+1)}\left(1+\frac{1}{\alpha}\right) + \frac{x_i}{\alpha^2}\, e^{-\lambda x_i}\left(e^{-\alpha\lambda x_i}-1\right)\right]\left[\frac{a-1}{1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})} - \frac{a(b-1)\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})\right)^{a}}\right]$$

$$\frac{\partial^2 \ell_{KWE}}{\partial \alpha\, \partial a} = \frac{\partial^2 \ell_{KWE}}{\partial a\, \partial \alpha} = \sum_{i=1}^{n}\frac{1}{\alpha}\, e^{-\lambda x_i}\left(\frac{1}{\alpha} - e^{-\lambda\alpha x_i}\left(\frac{1}{\alpha}+\lambda x_i\right)\right)\left[\frac{1}{1-\frac{1}{\alpha}e^{-\lambda x_i}(\alpha+1-e^{-\lambda\alpha x_i})} -\right.$$

$$\frac{(b-1)\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^{a}}\left(1+\left(a\ ln\left(1-\frac{1}{\alpha}\ e^{-\lambda x_i}\left(\alpha+1-\right.\right.\right.\right.$$

$$\left.\left.\left.\left.e^{-\lambda\alpha x_i}\right)\right)\right)\left(1+\frac{\left[1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right]^{a}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^{a}}\right)\right)\Bigg]$$

$$\frac{\partial^2\,\ell_{KWE}}{\partial\alpha\,\partial b}=\frac{\partial^2\ell_{KWE}}{\partial b\,\partial\alpha}=$$

$$-\sum_{i=1}^{n}\frac{1}{\alpha}\ e^{-\lambda\,x_i}\left(\frac{1}{\alpha}-e^{-\lambda\alpha\,x_i}\left(\frac{1}{\alpha}+\lambda x_i\right)\right)\left(\frac{a\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^{a}}\right)$$

$$\frac{\partial^2\ell_{KWE}}{\partial b\,\partial a}=\frac{\partial^2\,\ell_{KWE}}{\partial a\,\partial b}=-\sum_{i=1}^{n}\frac{\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^{a}ln\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^{a}}$$

$$\frac{\partial^2\ell_{KWE}}{\partial\lambda\,\partial b}=\frac{\partial^2\,\ell_{KWE}}{\partial b\,\partial\lambda}=$$

$$-\sum_{i=1}^{n}x_i\ e^{-\lambda x_i}\left(1+\frac{1}{\alpha}-e^{-\lambda\alpha x_i}\left(1+\frac{1}{\alpha}\right)\right)\left(\frac{a\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^{a}}\right)$$

$$\frac{\partial^2\ell_{KWE}}{\partial\lambda\,\partial a}=\frac{\partial^2\,\ell_{KWE}}{\partial a\,\partial\lambda}=$$

$$\sum_{i=1}^{n}x_i\ e^{-\lambda x_i}\left(1+\frac{1}{\alpha}-e^{-\lambda\alpha x_i}\left(1+\frac{1}{\alpha}\right)\right)\left[\frac{1}{1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)}-\right.$$

$$\frac{(b-1)\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^{a-1}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^{a}}\left(1+\left(a\ ln\left(1-\frac{1}{\alpha}\ e^{-\lambda x_i}\left(\alpha+1-\right.\right.\right.\right.$$

$$\left.\left.\left.\left.e^{-\lambda\alpha x_i}\right)\right)\right)\left(1+\frac{\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^{a}}{1-\left(1-\frac{1}{\alpha}e^{-\lambda x_i}\left(\alpha+1-e^{-\lambda\alpha x_i}\right)\right)^{a}}\right)\right)\Bigg]$$

When the convergence occurs between iteration $(h+1)$ and (h) , i.e. the absolute difference between two successive iterations is less than pre-specified error tolerance, $\varepsilon>0$, then the current $\hat{\alpha}^{(h+1)}$, $\hat{\lambda}^{(h+1)}$,$\hat{a}^{(h+1)}$ and $\hat{b}^{(h+1)}$ represent the MLEs of α ,λ ,a and b via NR algorithm which we referred to as, $\hat{\alpha}_{ML}$,$\hat{\lambda}_{ML}$,$\hat{a}_{ML}$ and $\hat{b}_{ML}$.

Then, according to an invariant property of the ML estimator, the estimate of reliability and hazard functions at mission time (t) can be obtained, respectively, by replacing α,λ,a and b in equations (16) and (17) by their ML estimates.

6. Concluding Remarks

The two parameter weighted exponential (WE) distribution introduced by Gupta and Kundu has been extension based on the family of Kumaraswamy generalized distribution introduced by Cordeiro and de Castro. Some of the mathematical properties along with the maximum likelihood estimations of the model parameters of new distribution named Kumaraswamy weighted exponential (KWE) have been discussed. The KWE density function can be expressed as an infinite linear combination of WE densities and the WE distribution is a special case of KWE distribution when $a = b = 1$ and the exponentiated WE distribution is a special case of KWE distribution when $b = 1$.

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