On some Properties of divisible Fuzzy subgroup

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Abstract :
The purpose of this paper is to study some of properties of group theory. Which we can apply of fuzzy subgroup and fuzzy divisible subgroup.
We serve some propositions, theory's and notes about fuzzy subgroup and fuzzy divisible subgroup and we set necessary sufficient condition carried over some properties of divisible fuzzy subgroup.

بعض الخواص للزمر الجزئية الضبابية القابلة للقسمة

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الخلاصة
يهدف اليحث الى دراسة بعض الخواص في نظرية الزمر والتي يمكن تطبيقها على الزمر الضبابية الجزئية، ولقد طرحت الباحثة بعض القضايا والنظريات حول الزمر الجزئية الضبابية والزمر الجزئية الضبابية القابلة للقسمة وقامت الباحثة بوضع الشروط الضرورية والكافية واللازمة من اجل تحقيق ذلك.

Introduction: Zadein in 1965 [10] introduced the concept of fuzzy set. Anthony and she rood [1] introduced the concept of fuzzy group, in this paper, we study certain properties of fuzzy subgroup. We notice that many basic properties in group theory carried over on fuzzy group.

1- Fuzzy set: In this section we shall start to introduce the concepts about fuzzy sets an the basic definitions with some examples, also we shall give some important definitions and properties with operation on fuzzy set which are used in the next.

Definition (1.1): [10]: Let A be anon- empty set, the function M: R→ [0,1] such that

$$M(x) = \begin{cases} 1-\frac{1}{x} & if\ x > 1 \\ 0 & if\ x \leq\ 1 \end{cases}$$ Is fuzzy set.

Definition (1.3): [3] Let $X_t: A \to [0,1], x \in A$ be a fuzzy set of A and $t \in [0,1]$ defined by $X_t(y)$ for all $y \in A$

$$Xt(y) = \begin{cases} t & if\ x = y \\ 0 & if\ x \neq y \end{cases}$$ Then X_t is called a fuzzy singleton.

Example (1.4) : Let $M_1: Z \to [0,1]$ such that.

$$M_1(x)=\begin{cases}\frac{1}{5} & if\ x\in z_e\\ \frac{1}{8} & if\ x\in z_o\end{cases}\quad ,M_2(x)=\begin{cases}\frac{1}{4} & if\ x\in z_e\\ \frac{1}{5} & if\ x\in z_o\end{cases}$$

Then $M_1\subset M_2$ where Z_e is even integer number and Z_o is odd integer number.

Definition (1.5): [8]: Let M_1,M_2 be two fuzzy subset of A then

i) $M_1=M_2$ iff $M_1(x)=M_2(x),\forall\ x\ \in A$

ii) $M_1\subseteq M_2$ iff $M_1(x)\le M_2(x),\forall x\in A.$

iii) if $M_1\subseteq M_2$ and there exists , $x\in A$ such that $M_1(x)<M_2(x)$,then we write $M_1\subset M_2(x)$

Definition (1.6): [9]: Let M_1,M_2 be two fuzzy set in A then.

i) $(M_1\cup M_2)(x)=\max\{M_1(x),M_2(x)\}$, for each $x\in A$

ii) $(M_1\cap M_2)(x)=\max\{M_1(x),M_2(x)\}$, for each $x\in A$

Notice that $(M_1\cup M_2)$ $(M_1\cap M_2)$ are fuzzy set in A.

If we generalize this definition by a collection of fuzzy sets then:

$(\cup_{a\in\Omega} M_1^{\alpha})(x)=\sup\{M_1^{\alpha}(x)\mid\alpha\in\Omega\}$, for each $x\in A$

$(\cap_{a\in\Omega} M_1^{\alpha})(x)=\sup\{M_1^{\alpha}(x)\mid\alpha\in\Omega\}$, for each $x\in A$

Which are also fuzzy sets in A.

Example (1.7): In the example (1.4)

If $x\in Ze\ then (M_1\cup M_2)(x)\max\{M_1(x),M_2(x)\}=\max\left\{\frac{1}{5},\frac{1}{4}\right\}=\frac{1}{4}$

If $x\in Ze\ then (M_1\cup M_2)(x)\max\{M_1(x),M_2(x)\}=\max\left\{\frac{1}{8},\frac{1}{5}\right\}=\frac{1}{5}$

Now If $x\in Ze\ then (M_1\cap M_2)(x)\max\{M_1(x),M_2(x)\}=\max\left\{\frac{1}{8},\frac{1}{5}\right\}=\frac{1}{8}$

If $x\in Ze\ then (M_1\cup M_2)(x)=\min\left\{\frac{1}{8},\frac{1}{5}\right\}=\frac{1}{8}$

Hence $(M_1\cap M_2)(x)=\begin{cases}\frac{1}{5}\ldots\ldots if\ x\in Z_e\\ \frac{1}{8}\ \ldots\ldots.if\ x\in Z_o\end{cases}$

Notes (1.8): 1) Denote to [0.1] by $\mathrm{I}=[0,1]$, 2) $\mathrm{I}^{\mathrm{A}}=\{M:A\to I\ fuzzy\ set\}$

2- Fuzzy subgroups of a Group : This section consists the concepts of the fuzzy groups which was coined by Rosenfeld [9] , who found many basic properties in group theory carried over on fuzzy group and in the same way applied to another algebraic structures like rings, ideals, modules and so on (see [6] , [2]).

Definition (2.1) [6] : Let (G, •) be a semi- group i.e. , • : G×G → G

Such that Im (•) ⊆ G and let M_1,M_2 ∈ IG

Then for each $x\in G$ we define:

$$(M_1,M_2)(x)=\begin{cases}\sup\{\min\{M_1(x_1),M_2(x_2)\}\ if\ x\in Im(\bullet)\\ 0\qquad otherwise\end{cases}$$

Its clear that, M_1,M_2 are fuzzy subsets of G. And if we take a collection $\{M\mid\alpha\ \in\Omega\}$ of fuzzy subsets then for each $x\in G$:

$$(\prod a\le\ \Omega\, M_{\alpha}(x)\begin{cases}\sup\{\inf\{M_{\alpha}\ (x_{\alpha})\mid\ =\prod a\le\ \Omega\in G\ \forall\alpha\ \in\Omega\\ 0\qquad otherwise\end{cases}$$

Proposition (2.2): [6] Let (G, •) be a semi- group X_t, Y_s be two fuzzy singletons, where $X, Y \in G$ and $t, s \in$ [0,1]. Then $xt., ys = (xy)$, where r=min$\{t, s\}$

$$(x_r, y_s)(z) = \begin{cases} \sup\ \{\min\{x_1(z_1), y_s(z_2) \mid and\ z_1.z_2 \in G\}\} & if\ x \in Im(\bullet) \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} \min\{t, s \mid x = z, and\ y = z_2\} = r & if\ z \in Im(\bullet) and z = x.y \\ 0 & otherwise \end{cases} \quad (1)$$

On the other hand

$$(\,(x.y)r)\,(..) = \begin{cases} & if\ z = xy \\ 0 & otherwise \end{cases} \quad (2)$$

From (1) and (2) we have $X_1 Y_s = (xy)_r$

Where $r = \min\{t, s\}$

Definition (2.3): [5] Let G be a non- empty set and closed a binary operation (•) and $M \in I^G$ such that $M \neq \emptyset$, where $\emptyset$ is the empty fuzzy set defined by $\emptyset(x) = 0$ for each $x \in G$. Then (M,•) is called closed if and only if $M\ M \subseteq M$.

Proposition (2.4): [6] Let $M \in I^G$ and $M \neq \emptyset$. Then the following statements are equivalent:

i) $(M, .) is\ closed$

ii) For any $X_t, Y_t \subseteq M$. Then $X_t, Y_t \subseteq M$ for each $x, y \in G$.

iii) $M(x, y) \geq \min\{M(x), M(y)\}$ for each $x, y \in G$.

Now we ready to define a fuzzy subgroup of a group.

Definition (2.5): [2] Let (G,.) be a group and $M \in I^G$ such that $M \neq \emptyset\,.(M(x) \neq O\ \forall x \in G)$. Then M is called a fuzzy subgroup of G if and only if for each $X, Y \in G$.

1) $M(x, y) \geq \min\{M(x), M(y)\}$, 2) $M(x) = M(x^{-1})$

Proposition (2.6) : [4] Let M_1 and M_2 is a fuzzy subgroup of G and $n \in N$. Then $M_1 + M_2$ is a fuzzy subgroup of G.

Definition (2.7): [7] Let G be a group and M be a fuzzy subgroup of G. then we define the following:

1) $M^* = \{x \in G \mid M(x) > 0\}$ is called the support of M. also $M^* = \underset{t \in (0,1]}{\cup} M_t$

2) $M_* = \{x \in G \mid M(x) = M(e)\}$ it is easy to show that M^* and M_* are subgroups of G.

Definition (2.8): [1] A fuzzy subset of a group (G, +) is called has the supremum property iff sup $\{M(x) \mid y = f(x)\} = \max\{M(x) \mid y = f(x)\}$, where f is a function from G to G.

Definition (2.9): [7]

Let M_1 and M_2 is a fuzzy subgroup of G, then the intersection is a fuzzy subgroup of G, by their intersection $M_1 \cap M_2$ is denoted by:

$(M_1 \cap M_2)(x) = \min\{M_1(x), M_2(x)\}$ for all $x \in G$

Proposition (2.10) [4] : Let M_1, M_2 be two fuzzy subgroup of G then $M_1 \cap M_2$ is fuzzy subgroup.

Proof: Let $x, y \in G$

$(M_1 \cap M_2)(x, y) = \min\{M_1(x, y), (M_2(x, y)\}$

$\geq \min\{\min\{M_1, (x), M_1(y)\} \min\{M_2(x), M_2(y)\}$

$= \min\{M_1(x), M_1(y), M_2(x), M_2(y)\}$
$= \min\{\min\{M_1(x), M_2(y)\}, \min\{M_1(x), M_2(y)\}\}$
$= \min\{M_1 \cap M_2)(x), (M_1 \cap M_2)(y)\}\}$
$(M_1 \cap M_2)(x) = \min\{M_1(x), M_2(x)\}$
$= \min\{M_1(-x), M_2(-x)\} \quad = (M_1 \cap M_2)(-x)$

Remark (2,11) [ ]: The union of two fuzzy subgroup are not need to be a fuzzy subgroup as illustrated in the following example.

Example (2.12): Let G be aklein`s four group.
$\mathrm{G} = \{\mathrm{e}, \mathrm{f}, \mathrm{g}, \mathrm{fg}\}$, where $\mathrm{f}^2 = \mathrm{e} = \mathrm{g}^2$ and $\mathrm{fg} = \mathrm{gf}$.
For $0 \le i \le 5$, $let\ t_i \in [0,1] such\ that\ 1 = t_o > t_1 > \cdots > t_5, define\ fuzzy$ Subsets $M_1 and\ M_2: G \to [0,1] as\ follows.$
$\mathrm{M}_1(\mathrm{e}) = \mathrm{t}_1$, $\mathrm{M}_1(\mathrm{f}) = \mathrm{t}_3$, and $\mathrm{M}_1(\mathrm{g}) = \mathrm{m}_1(\mathrm{fe}) = \mathrm{t}_4$, $\mathrm{M}_2(\mathrm{e}) = \mathrm{t}_0$, $\mathrm{M}_2(\mathrm{f}) = \mathrm{t}_5$ $\mathrm{M}_2(\mathrm{g}) = \mathrm{t}_2$ and $\mathrm{M}_2(\mathrm{fg})$
$= \mathrm{t}_5$, it can we seen that M_1 and M_2 are fuzzy subgroup of G.

Proposition (2.13) [4]: Let M_1 and M_2 be two fuzzy subgroup of G and $M_1 \subseteq M_2 \subseteq M_1$ then $M_1 \cup M_2$ is fuzzy subgroup of G.

Proof: suppose $M_1 \subseteq M_2$
Then $(M_1 \cup M_2)(x) = \max\{M_1(x), M_2(x)\} = M_2(x) \forall x \in G.$
Let $x, y \in G.$
$(M_1 \cup M_2)(xy) = \max\{xy), M_2(xy)\}$
$= M_2(xy) \ge \min\{M_2(x), M_2(y)\}$ (1)
$\min\{(M_1 \cup M_2)(x), (M_1 \cup M_2)(y)\}$
$= \min\{\max\{M_1(x), M_2(x)\}, \max(M_1(y), M_2(y)\}$
$= \min(M_2(x), M_2(y)\}$ (2)
From (1) and (2)
$(M_1 \cup M_2)(xy) \ge \min\{(M_1 \cup M_2)(x), (M_1 \cup M_2)(y)\}$
$\therefore M_1 \cup M_2$ fuzzy subgroup.

Definition (2.15): [7] Let M_1 and M_2 are fuzzy subgroups of group G. The sum of M_1 and M_2 over a group G is denoted by:
Sup
$\{\min\{M_1(x_1), M_2(x_2) \mid \mathrm{x} = \mathrm{x}_1, x_2, x_1, x_2 \in G, if\ x \in I\,m(.)\}\}\ is\ a\ \ subset\ of\ G.$

Definition (2.16): [7]
Let (G, +) be a group and M be a fuzzy subset of G and let P aprime number. Define a fuzzy subset PM of G by:

$$(PM)(x) = \begin{cases} \sup\{M(y) \mid \mathrm{x} = \mathrm{py} & \text{if } \mathrm{x} \in \mathrm{PG} \\ 0 & if\ x \notin PG \end{cases}$$

Where PG= $\{Py = y + y + \cdots + y \mid \mathrm{y} \in \mathrm{G}\} \subseteq G.$
Now we are ready to give the following propositions.

Proposition (2.18) Let M_1 and M_2 be two fuzzy subgroups of G and P aprime number then:
i) pM is a fuzzy subgroup of G.
ii) $pM_t \subseteq (pM)_t, \forall t \in (0,1]$
iii) if M has supremum property then $pM_t \subseteq (pM)_t, \forall t \in (0,1]$
iv) $pM \subseteq M$

proof: Lett p aprime number prove pM is a fuzzy subgroup of G we must hold the following conditions :

i) For each x,y ∈

1) $PM(x) = pM(-x)$

$$(PM)(x) = \begin{cases} \sup\{M(u) \mid \text{x} = \text{pu}\} & \text{if x} \in \text{PG} \\ 0 & if\ x \notin PG \end{cases}$$

$$= \begin{cases} \sup\{M(-u) \mid \text{x} = \text{p}(-\text{u})\} & \text{if x} \in \text{PG} \\ 0 & if\ x \notin PG \end{cases} = PM(-x)$$

Hence PM (x)=pM(-x)

2) $(pM)(x + y) \geq \min\{pM(x), pM(y)\}$

$$(PM)(x) = \begin{cases} \sup\{M(u) \mid \text{x} = \text{pu}\} & \text{if x} \in \text{PG} \\ 0 & if\ x \notin PG \end{cases}$$

Also

$$(PM)(x) = \begin{cases} \sup\{M(v) \mid \text{x} = \text{pv}\} & \text{if x} \in \text{PG} \\ 0 & if\ x \notin PG \end{cases}$$

$$x = pu\ and\ y = pv\ implies\ that\ x + y = pu + pv = p(u + v) = pw$$

Therefore

$$(PM)(x) = \begin{cases} \sup\{M(w) \mid \text{x} = \text{pw}\} & \text{if x} \in \text{PG} \\ 0 & if\ x \notin PG \end{cases}$$

$$pM(x + y) =$$

$$\begin{cases} \sup\{M(u + v) \mid \text{x} + \text{y} = \text{pw}\} & \text{if x} \in \text{PG} \\ 0 & if\ x \notin PG \end{cases}$$

$If\ x + y \in pG\ , x \in pG\ and\ y \in pG, then$:

$pM(x + y) \geq \sup\{\min\{pM(u), pM(v) \mid x + y = p(u + v)\}$

$\geq \min\{pM(u), pM(v) \mid x = pu\}, \{\sup\{pM(u) \mid x = pu\}\}$

$\geq \min\{\sup\{pM(u) \mid x = pu\}, \{\sup\{pM(u) \mid x = pu,\}\}$

$= \min\{pM(x), PM(y)\}$

If $x + y \notin pG$, $x \notin pG$ and $y \notin pG$ then:

$pM(x + y) = \min\{pM(x), pM(y)\}$

Hence , $pM(x + y) \geq \min\{pM(x), pM(y)\}$

Therefore, pM is a fuzzy subgroup of G.

ii) let $x \in pM_t$ $.Then\ x = pw\ for\ some\ w \in M_t and\ M(w) \geq t.Thus$ $PM(x) = \sup\{M(w)|x = pw\} \geq t. To\ prove\ (pM)_t$

iii) From part (ii), $pM_t \subseteq (pM)_t .To\ prove\ (pM)_t \subseteq pM_t$

Let $\text{x} \in \text{PM}(\text{x}) = \sup\{\text{M}(\text{y})|\text{x} = \text{py}\} \geq \text{t}$ and since M has suprimum Property So $\exists\ y_o \in G$ such that $x = py_o$ and $M(y_o) =$ $\sup\{M(y)| \geq t$. Therefore, $y_o \in M_t$ and then $py_o = x \in$ pM_t. i. e $(\text{pM})_\text{t} \subseteq (\text{pM})_\text{t}$

Hence $(\text{pM})_\text{t} \subseteq (\text{pM})_\text{t}$

iv) $\forall\ x\ \in G\ and\ P$ aprime number.

We have $PM(x) = \sup\{M(u)|x = pu\}$ Nw $\forall u \in G\ such\ that\ x = pu$ Implies $M(pu) \geq M(u)$. Therefore, $M(x) \geq \sup\{M(u)|x = pu\} =$ $(pM)(x). thus\ PM \subseteq M.$

Proposition (2.18) Let M be fuzzy subgroup of G, then for each a prime numbers $p_1, p_2, \dots, p_n$. $(p_1, p_2 \dots., p_n)M$ is a fuzzy subgroup of G.

Proof: Let $p_1, p_2 \ldots., p_n$ a prime numbers prove $(p_1, p_2, \ldots. p_n)M$ is a fuzzy subgroup of G we must prove the following conditions:

For each $x, y \in G$.

1) $(p_1, p_2, \ldots., p_n)M(x) =$
$$\begin{cases} \sup[\mathrm{M(u)}|\mathrm{x} = (\mathrm{p_1.p_n} \ldots.., \mathrm{p_n})\mathrm{u}, \text{if } \mathrm{x} \in (\mathrm{p_1.p_2} \ldots. \mathrm{p_n G}\} \\ 0 \qquad if\ x \notin (p_1.p_2 \ldots..p_n)G \end{cases}$$

2) $(p_1, p_2, \ldots., p_n)M(x) =$
$$\begin{cases} \sup[\mathrm{M(-u)}|\mathrm{x} = (\mathrm{p_1.p_n} \ldots.., \mathrm{p_n}) - \mathrm{u}, \text{if} - \mathrm{x} \in (\mathrm{p_1.p_2} \ldots. \mathrm{p_n G}\} \\ 0 \qquad if - x \notin (p_1.p_2 \ldots..p_n)G \end{cases}$$
$= (p_1.p_2 \ldots..p_n)M(x)$

Hence $(p_1, p_2, \ldots\ldots, p_n)M(x) = (p_1.p_2, \ldots..p_n)M(-x)$

2) $(p_1.p_2 \ldots\ldots p_n)M(x+y)$
$$\geq \min\{(p_1.p_2 \ldots\ldots p_n)M(x)$$
$$= \begin{cases} \sup[\mathrm{M(v)}|\mathrm{x} = (\mathrm{p_1.p_n} \ldots.., \mathrm{p_n})\mathrm{v}, \text{if } \mathrm{x} \in (\mathrm{p_1.p_2} \ldots. \mathrm{p_n G}\} \\ 0 \qquad if\ x \notin (p_1.p_2 \ldots..p_n)G \end{cases}$$

$x = (p_1.p_2 \ldots.p_n)u\ and\ y = (p_1.p_2 \ldots.p_n)v$ implies that:

$x + y = (p_1.p_2 \ldots..p_n)u + (p_1.p_2 \ldots.p_n)v$

$= (p_1.p_2 \ldots..p_n)uv = (p_1.p_2 \ldots..p_n)w$

$Therefore$

$: (p_1.p_2 \ldots\ldots p_n)$

$M(x + y) = \begin{cases} \sup[\mathrm{M(u+v)}|\mathrm{x+y} = (\mathrm{p_1.p_2} \ldots.., \mathrm{p_n})(\mathrm{u}, \text{if } \mathrm{x} \in (\mathrm{p_1.p_2} \ldots. \mathrm{p_n G}\} \\ 0 \qquad if\ x \notin (p_1.p_2 \ldots..p_n)G \end{cases}$

$(p_1.p_2 \ldots\ldots p_n)M(x + y) = \begin{cases} \sup[\mathrm{M(u+v)}|\mathrm{x+y} = (\mathrm{p_1.p_2} \ldots.., \mathrm{p_n})(\mathrm{u+v}), \text{if } \mathrm{x} \in (\mathrm{p_1.p_2} \ldots. \mathrm{p_n G}\} \\ 0 \qquad if\ x \notin (p_1.p_2 \ldots..p_n)G \end{cases}$

$If\ x + y \in (p_1.p_2 \ldots.p_n)G, x \in (p_1.p_2 \ldots..p_n)G, y \in (p_1.p_2 \ldots..p_n)G$

$(p_1.p_2 \ldots.p_n)M(x + y) \geq$

$\sup\{\min\{M, M(v)|x + y = (p_1.p_2 \ldots.p_n)M(u + v)\}\}$

$\geq \min\{M(u), (v)|x + y = (p_1, p_2 \ldots p_n)u\} \sup\{\{M(u + v)\}$

$\geq \min\{\sup\{M(u)|x = (p_1.p_2 \ldots.p_n)u\} \sup\{\{M(v)|y = (p_1.p_2 \ldots p_n\}v\}\}$

$If\ x + y \notin (p_1.p_2 \ldots.p_n)G, x \notin (p_1.p_2 \ldots.p_n)G, y \notin (p_1.p_2 \ldots p_n)G$

$(p_1.p_2 \ldots p_n)M(x + y) = \min\{(p_1.p_2 \ldots p_n)M(x), (p_1.p_2 \ldots p_n)M(y)\}$

Hence

$(p_1.p_2 \ldots p_n)M(x + y) \geq \min\{(p_1.p_2 \ldots p_n)M(x), (p_1.p_2 \ldots p_n)M(y)\}$

Therefore $(p_1.p_2 \ldots p_n)M$ is fuzzy subgroup of G.

Proposition (2.19): Let M be fuzzy subgroup of G, then for each a prime numbers P and n any positive integers p^n M is a fuzzy subgroup of G.

Proof: To prove p^n M is a fuzzy subgroup of G we must prove the following conditions:

For each x,y $\in G$

$p^n M(x) = p^n M(-x)$

$$p^n M(x) = \begin{cases} \sup\{M(u)|x = p^n u\}\ if\ x \in p^n G \\ 0 \qquad if\ x \notin p^n G \end{cases}$$

$$= \begin{cases} \sup\{M(-u) | x = p^n(-u)\} \; if \; x \in p^n G \\ 0 \qquad if - x \notin p^n G \end{cases}$$

$= p^n M(-x)$

Hence $p^n M(x) = p^n M(-x)$

2) $(p^n M)(x + y) \geq \min\{p^n M(x), p^n M(y)\}$

$$p^n M(x) = \begin{cases} \sup\{M(u) | x = p^n u\} \; if \; x \in p^n G \\ 0 \qquad if \; x \notin p^n G \end{cases}$$

Also

$$p^n M(x) = \begin{cases} \sup\{M(v) | x = p^n v\} \; if \; x \in p^n G \\ 0 \qquad if \; x \notin p^n G \end{cases}$$

$x = p^n u \; and \; y = p^n v \; implies \; that \; x + y = p^n u + p^n v = p^n (u + v)$
$= p^n w$

$p^n w$

Therefore

$p^n M(x + y) =$

$$\begin{cases} \sup\{M(w) | x + y = p^n w\} \; if \; x \in p^n G \\ 0 \qquad if \; x \notin p^n G \end{cases}$$

$p^n M(x + y) =$

$$\begin{cases} \sup\{M(u + v) | x + y = p^n (u + v)\} \; if \; x \in p^n G \\ 0 \qquad if \; x \notin p^n G \end{cases}$$

$If \; x + y \in p^n G, x \in p^n G \; and \; y \in p^n G, then:$

$p^n M(x + y) \geq \sup\{\min\{p^n M(u), p^n M(v) | x + y = p^n (u + v)\}$

$\geq \min\{p^n M(u), p^n M(v) | x + y = p^n (u + v)\}$

$\geq \min\{\sup\{p^n M(u) | x = p^n u\}, (\sup(p^n M(u) | x = p^n u\} \}$

$= \min\{p^n M(x), p^n M(y) \}$

$If x + y \notin p^n G, x \notin p6n \; G \; and \; y \notin p^n G, then:$

$p^n M(x + y) \min\{p^n M(x), p^n M(y)\}$

Hence $p^n M(x + y) \geq \min\{p^n M(x), p^n M(y)\}$

Therefore $p^n M$ is fuzzy subgroup of G.

Proposition (2.20): Let M_1 and M_2 be two fuzzy subgroups of G. Then (M_1, M_2) is a fuzzy subgroups of G.

Proof: To prove (M_1, M_2) is a fuzzy subgroups of G we need to satisfy two conditions:

For each $x, y \in G$

1) $(M_1, M_2)(x) = \sup\{\min\{M_1(x_1), M_2(x_2)\} | x = x_1 . x_2, x_1, x_2 \in G\}$

$= \sup\{\min\{M_1(-x_1), M_2(-x_2)\} | -x = -x_1 . -x_2, x_1, x_2 \in G\}$

$= \sup\{\min\{M_1(x_1), M_2(x_2)\} | -x = x_1 . x_2, x_1, x_2 \in G\}$

$= (M_1 . M_2)(-x)$

2) $(M_1, M_2)(x) = \sup\{\min\{M_1 . M_2\}(x), (M_1 . M_2)(y)\}$

$\forall x, y \in G. Let \; x = x_1 . x_2 (\forall \; x_1, x_2 \in G), y = y_1 . y_2 (\forall \; y_1, y_2 \in G)$

Then $: x . y = (x_1 . x_2) . (y_1 . y_2)$ since G is commutative group.

So $x . y = (x_1 . x_2) . (y_1 . y_2) = u . v.$

Now we have:

$(M_1 . M_2)(x, y) = \sup\{\min\{M_1(u), M_2(v) | x . y = u . v, u, v \in G\}$

$\geq \min\{M_1(x_1 . y_1), M_2(x_2 . y_2)\} | x . y = (x_1 . y_1) . (x_2 . y_2)\}$

$\geq \min\{M_1(x_1), M_1(y_1) \min\{M_2(x_2), M_2(y_2)\}\}$
$= \min\{M_1(x_1), M_1(y_1)\ , M_2(x_2), M_2(y_2)\}$
$\min\{\min\{M_1(x_1), M_2(x_2)\} x = x_1 . x_2 \min\{M_1(y_1), M_2(y_2)\} | y = y_1 . y_2\}\}$
Thus $(M_1 . M_2)(x . y) \geq \sup\{\min\{M_1(x_1), M_2(x_2)\} | x = x_1 . x_2\}$
$\sup\{\min\{M_1(y_1), M_2(y_2)\} | y = y_1 . y_2\}\}$
$= \min\{(M_1 . M_2)(x), (M_1 . M_2)(x)\}$
Hence ,$(M_1 . M_2) \geq \min\{M_1 . M_2)(x), (M_1 . m_2)(y)\}$
Therefore , $(M_1 . M_2)$ is a fuzzy subgroup of G.

3) Divisible fuzzy subgroup of an abelian group: In this section we will introduce a fuzzy subgroup of an abelian group G which called divisible fuzzy subgroup and give some properties.

Definition (3.1): [7] M is called a divisible fuzzy subgroup of an abelian group G iff M is a fuzzy subgroup and for each fuzzy singleton $x_t \subseteq M\ with\ t > 0$ and for each $n \in N$ there exists a fuzzy singleteon $y_t \subseteq M$ such that $n(y_t) = x_t$.

Propositions (3.2): From definition [2.5] M is a fuzzy subgroup iff M_t is a fuzzy subgroup of G.
M be divisible fuzzy subgroup iff $\forall\ x_t \subseteq M\ , t > 0, \forall n \in N.$
$\exists\ y_t \subseteq M\ such\ that\ n\ (y_t) = x_t$
$n(y_t) = x_t iff\ (ny)_t = x_t iff\ ny = x$
Hence
$(\forall\ x\ \in M_t, \forall n\ \in N), \exists y\ \in M_t such\ that\ ny =$
$x\ iff\ m_t is\ divisible\ fuzzy\ subgroup\ of\ G.$

Theorem (3.3): [7]
1) If M is divisible fuzzy subgroup, then M^* subgroup of G.
2) If M is divisible fuzzy subgroup, then M_* is divisible fuzzy subgroup of G.

Theorem (3.4): [7] If M^* id divisible fuzzy subgroup and M is a constant on $M^* - \{0\}$, then M is divisible fuzzy subgroup.
In ordinary group , G is divisible fuzzy subgroup iff nG=G, $\forall\ n \in N$
But this property is not valid in fuzzy subgroup as we see in the fowling theorem.

Theorem (3.5): If M is a fuzzy subgroup of G, then:
1) $nM \subseteq M\ , \forall\ n\ \in N.$
2) $If\ M\ is$ divisible fuzzy subgroup, then $nM = M\ , \forall n \in N.$
1) If M has the supremum property and $nM = M, \forall n \in N\ then\ M\ is$ divisible fuzzy subgroup of G.

Proof:
1) $\forall\ x\ \in G\ and\ n\ \in N$ we have $(nM)(x) = \sup\{M(u) | x = nu\}$
Now for each $n\ \in G$ such that $x = nu$ implies $M(x) = M(nu) \geq M(u)$
Therefore $M(x) \geq \sup\{M(u) | x = nu\} = (nM)(x)$
Thus $nM \subseteq M.$
2) Suppose M is divisible fuzy subgroup, by part (1) above we have $nM \subseteq M, \forall\ n\ \in N.$ Now let $M(x) = t$ with $t > 0$ Since M is divisible fuzzy subgroup, then $\forall n \in N, \exists\, y_t \subseteq M$ such that $n(y_t) = x_t.$
Since , $\forall\ y_t \subseteq M\ so\ M(y) \geq t.$ Hence , sup $\{M(y) | x = ny\} \geq t = M(x)$

This implies $(nM)(x) \geq M(x), \forall x \in G\ then\ M \subseteq nM, \forall n \in N$.
Therefore $nM \subseteq M, \forall n \in N$.
3) Since $nM \subseteq \forall n \in N\ then\ (nM)(x) = M(x), \forall x \in G$
Thus there exists, $\forall x \in G$ thus that $nM(x) = \sup\{M(y) | x = ny\}$
Since M has the supremum property, then $\exists\ y_o \in G$ such that $nM(x) = \sup\{M(y) | x = ny\} = M(y_o) where\ x = ny_0$
Let $x_t \subseteq M$ Now, $(nM)(x) = M(y_o) = M(x) \geq t$ therefore $y_o \in M_t and\ ny = x$
This implies $(y_o)_t \subseteq M$ and $n(y_o)_t = x$
Hence , M divisible fuzzy subgroup of G.

Proposition (3.6) : If M is a fuzzy subgroup of G , then for each prime number p we have
1) If M is divisible by p, then pM=M.
2) If M has the supremum property and pM=M then M is divisible by p.

Proof:
1) $\forall p \in N\ and\ \forall t \in (0,1] we\ have$
M is divisible fuzzy subgroup $\Longrightarrow M_t$ is divisible fuzzy subgroup.
$\Longrightarrow pM_t = M_t$ by proposition (2,17) part ii we have $pM_t \subseteq (pM)_t$
$\Longrightarrow pM_t \subseteq (pM)_t$
$\Longrightarrow M \subseteq pM\ and\ by\ theorm\ (3,5), pM \subseteq M$
Hence $pM = M$
2) $\forall\ p \in N\ and\ \forall\ t \in (0,1] we\ have$:
$pM = M \Longrightarrow pM_t = M_t, by\ propostion\ (3,2)$
Then $pM_t = \mathrm{M_t}$ hence $\mathrm{M_t}$is divisible fuzzy subgroup of G.
$\Longrightarrow M$ is divisible fuzzy subgroup of G.

Theorem (3.7): Let $M_1 and\ M_2$ are two divisible fuzzy subgroups of G. then $M_1 + M_2$ is divisible fuzzy subgroup of G.

Proof: From proposition (2,6) $M_1 + M_2$ be fuzzy subgroup of G. $M - 1 + M_2$ be divisible fuzzy subgroup iff $x_i + g_j \subseteq M_1 + M_2, i, j > 0\ \forall\ n \in N, (x_i \subseteq M_1, g_j \subseteq M_2, i\ j > 0\ \forall\ n \in N\ since\ M_1, M_2$ are divisible fuzzy subgroups).
$\exists y_i + i_J \subseteq M_1 + M_2 such\ that\ n\ (y_i + l_j) = x_i + g_j, y_i \subseteq M_1, l_J \subseteq M_2\ , i\ , j > 0, \forall\ n \in N\ since\ M_1, M_2$ are divisible fuzzy subgroups).
$(y_i + l_j) = x_i + g_j\ iff\ n(y + l)_t = (x + g)_t where\ (I + j = t) iff\ n(y + 1) = x + g.$
Hence $\forall x, g\ \in (M_1 + M_2)_t\ \forall n\ \in N\ , \exists\ y + l\ \in (M_1 + M_2)_t such\ that\ n(y + 1) = x + g\ iff\ (M_1 + M_2)_t$ is divisible fuzzy subgroup of G.

Theorem (3.8): If $M_1 + M_2 be$ divisible fuzzy subgroups of G then for each prime number P we have:
1) If $M_1 + M_2$ be divisible by p, then $p(M_1 + M_2) = M_1 + M_2$
2) If $M_1 + M_2$ has the supremum property and $p(M_1 + M_2) = M_1 + M_2$
Then $M_1 + M_2$ be divisible by p.

Proof:
1) Let $M_1 + M_2$ is divisible fuzzy subgroup, by theorem (3,7)
We have $p(M_1 + M_1) \subseteq M_1 + M_2, \forall\ p \in N$.
Now , suppose $(M_1 + M_2) = t, t > 0\ since\ M_1 + M_2 be\ divisible$

then $\forall\ n \in N, \exists\ (y+1)_t \subseteq M_1 + m - 2\ \ such\ that\ p(y+l)_t = (x+g)_t$
since $(y+l)_t \subseteq M\ and\ M(y+l) \geq t$
Hence , $\{\sup\{M_1 + M_2)(y+l) | x + g = p(y+l)\} \geq t = (M_1 + M_2)(x+g)$
This implies $p(M_1 + M_2)(x+g) \subseteq (M_1 + M_2(x+g), \forall\ x + g\ \in G.$
Then $M_1 + M_2\ \subseteq p(M_1 + M_2), \forall p \in N.$
$Hence\ p(M_1 + M_2) = M_1 + m - 2$
2) $since\ p(M_1 + M_1) \subseteq M_1 + M_2, \forall\ p \in N, then\ p(M_1 + M_2)(x+g) \subseteq (M_1 + M_2)(x+g), \forall x + y \in G. thus\ ther\ exists\ y + l\ \in G\ such\ that.$
$(M_1 + M_2)(x+y) = \sup\{M(y+l) | x + g = p(y+l)\}$ sin $M_1 + M_2$ has the supremum property, then $\exists\ (y+l)_o \in G$ such that.
$pM(x+y) = \sup\{M(y+l)\ | x + g = p(y+l) = M(y+l)_o\}$
Where $x + g = p(y+l)_o$ Now $p(M_1 + M_2)_{(x+g)} = (M_1 + M_2)_{(y+g)_0} = (M_1 + M_2)_{(x+g)} \geq t\ , therefore\ (y+l)_o \in (M_1 + M_2)_t$ and $p(y+l)_o = x + g$
Hence $M_1 + M_2$ be divisible fuzzy subgroup of G.

Preposition (3.9): If M is fuzzy subgroup of G, p is a prime number and $n\ \in N\ the$:
1) IF M be divisible by p^n then $p^n\ M = M$
2) If M be divisible by $(p_1, pa_2, \dots. p_n)$ then $(p_1, p_2, \dots p_n)M = M$
3) If M has the supremum property and $p^n M = M$ then M is divisible by p^n
4) If M has the supremum property and $(p_1, p_2, \dots. p_n)M = M$ then M is divisible by $(p_1, p_2, \dots.. p_n)$

Proof: The prove of this proposition by putting p^n and $(p_1, p_2, \dots. p_n)$ instead of the positive integer (n) in theorem (3,5) part 2,3.

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